2. L. I. Rozein and T. I. Abramov, "Thermal calculations on current leads for cryogenic electrical engineering devices," Inzh.-Fiz. Zh., 20, No. 2 (1971).
3. A. G. Shifel'bain and S. P. Gorbachev, "Thermodynamic optimization of gas-cooled current leads with adjustable coolant flow," Cryogenic Engineering: Processes in Equipments and Systems [in Russian], Tr. NPO Kriogenmash, Mashinostroenie, Moscow, Issue 17 (1975).
4. A. G. Shifel'bain, S. P. Gorbachev, and V. K. Matyushchenkov, "Features of compound current leads," in: Superconductivity [in Russian], Vol. 5, Atomizdat, Moscow (1977).
5. A. I. Bertinov and S. V. Odenov, "Current leads for cryogenic devices," in: Superconductivity [in Russian], Vol. 5, Atomizdat, Moscow (1977).
6. M. P. Malkov (editor), Handbook on the Technical Physics of Cryogenics [in Russian], Énergiya, Moscow (1973).

ESTIMATING THE RANGE OF APPLICABILITY OF THE HYPERBOLIC THERMAL CONDUCTIVITY EQUATION

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A generalized thermal conductivity equation is considered. The geometric dimensions of regions in which temperature fields may be described by hyperbolic or parabolic thermal conductivity equations are estimated.

In the last decade wide use has been made of the hyperbolic thermal conductivity equation

$$\tau_r \frac{\partial^2 T(x,\ \tau)}{\partial \tau^2} + \frac{\partial T(x,\ \tau)}{\partial \tau} = a\ \frac{\partial^2 T(x,\ \tau)}{\partial x^2} \tag{1}$$

for description of high-intensity processes. In this equation, proposed in [1], τ_r is the relaxation time; a, thermal diffusivity coefficient; $W = \sqrt{a/\tau_r}$, λ, c, ρ, rate of propagation of heat, the thermal conductivity, the specific heat, and density.

In a layer of material of thickness l we will consider the mathematical model of the thermal conductivity process described by Eq. (1) with initial conditions

$$T(x,\ 0) = \varphi_1(x), \quad \frac{\partial T(x,\ 0)}{\partial \tau} = \varphi_2(x) \tag{2}$$

and boundary conditions

$$\alpha_{i1} \frac{\partial T((i-1)\,l,\ \tau)}{\partial x} + (-1)^i \alpha_{i2} T((i-1)\,l,\ \tau) = \varphi_{2+i}(\tau),\ i = 1,\ 2. \tag{3}$$

The coefficients α_{i1}, α_{i2} take on the values 0 and 1, depending upon the form of the boundary conditions.

Following [2], we construct the solution of the system (1)-(3) in the form

$$T(x,\ \tau) = \sum_{n=1}^{\infty} A_n(\tau) X_n(x) + \Psi(x,\ \tau). \tag{4}$$

An auxiliary, sufficiently smooth function $\Psi(x, \tau)$ which reduces inhomogeneous conditions to homogeneous is constructed in a manner such that

$$\Psi(x,\ 0) = \varphi_1(x), \quad \frac{\partial \Psi(x,\ 0)}{\partial \tau} = \varphi_2(x),$$

$$\alpha_{i1} \frac{\partial \Psi((i-1)\,l,\ \tau)}{\partial x} + (-1)^i \alpha_{i2} \Psi((i-1)\,l,\ \tau) = \varphi_{2+i}(\tau),$$

$$F((i-1)\,l,\ \tau) = 0, \quad i = 1,\ 2,$$

Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 37, No. 2, pp. 366-370, August, 1979. Original article submitted May 3, 1979.

 0022-0841/79/3702-0996$07.50

where

$$F(x,\ \tau)=a\frac{\partial^2\Psi(x,\ \tau)}{\partial x^2}-\frac{\partial\Psi(x,\ \tau)}{\partial\tau}-\tau_r\frac{\partial^2\Psi(x,\ \tau)}{\partial\tau^2}.$$

The functions $\{X_n(x)\}_{n=1}^{\infty}$ are normalized eigenfunctions of the following spectral problem:

$$\frac{d^2X(x)}{dx^2}+\mu X(x)=0,\quad \alpha_{i1}\frac{\partial X((i-1)\,l)}{\partial x}+(-1)^i\alpha_{i2}X((i-1)\,l)=0,\quad i=1,\ 2. \tag{5}$$

The coefficients $A_n(\tau)$ are defined with the aid of the infinite system

$$\tau_r\frac{d^2A_n(\tau)}{\partial\tau^2}+\frac{dA_n(\tau)}{d\tau}+\mu_n^2aA_n(\tau)=q_n(\tau),$$

$$A_n(0)=\frac{dA_n(0)}{d\tau}=0,\quad n=1,\ 2,\ \ldots, \tag{6}$$

$$q_n(\tau)=\int_0^l F(x,\ \tau)\,X_n(x)\,dx,$$

where μ_n are eigenfunctions of problem (5).

We write the characteristic equation for Eq. (6):

$$\nu^2+\tau_r^{-1}\nu+\mu_n^2a\tau_r^{-1}=0, \tag{7}$$

the solution of which has the form

$$\nu_1=-\frac{1+\delta_n^{1/2}}{2\tau_r},\quad \nu_2=-\frac{1-\delta_n^{1/2}}{2\tau_r},$$

$$\delta_n=1-4\mu_n^2a\tau_r=1-4\mu_n^2a^2W^{-2}. \tag{8}$$

Using [3], we obtain the solution of system (6):

$$A_n(\tau)=\begin{cases}\displaystyle\int_0^\tau q_n(s)\,\delta_n^{-1/2}\{e^{\nu_1(\tau-s)}-e^{\nu_2(\tau-s)}\}\,ds & \text{for}\ \ \delta_n>0,\\[2ex] \displaystyle\int_0^\tau q_n(s)\,\frac{t-s}{\tau_r}\exp\left\{-\frac{\tau-s}{2\tau_r}\right\}ds & \text{for}\ \ \delta_n=0,\\[2ex] \displaystyle\int_0^\tau q_n(s)\,2\,|\delta_n|^{1/2}\sin\frac{|\delta_n|^{1/2}(\tau-s)}{2\tau_r}\,ds & \text{for}\ \ \delta_n<0.\end{cases} \tag{9}$$

Then, in general form, we can write the solution of Eqs. (1)-(3) as

$$T(x,\ \tau)=R_p(x,\ \tau)+R_0(x,\ \tau)+R_h(x,\ \tau)+R_{m_0}(x,\ \tau)+\Psi(x,\ \tau),$$

where

$$R_p(x,\ \tau)=\sum_{n=1}^{n_0-1}A_n(\tau)\,X_n(x)\ \text{for all n for which}\ \ \delta_n>0,$$

$$R_0(x,\ \tau)=A_{n_0}(\tau)\,X_{n_0}(x)\ \ \text{if there exists an}\ \ n_0\ \text{such that}\ \ \delta_{n_0}=0,$$

$$R_h(x,\ \tau)=\sum_{n=n_0+1}^{m_0}A_n(\tau)\,X_n(x)\ \ \text{for all}\ \ \ n\leqslant m_0\ \ \text{for which}\ \ \delta_n<0.$$

We will term R_p, R_0, R_h the parabolic, transitional, and hyperbolic components of the solution of Eqs. (1)-(3), while

$$R_{m_0}(x,\ \tau)=\sum_{n=m_0+1}^{\infty}A_n(\tau)\,X_n(x)$$

is the residue of the series, with the number m_0 being determined from the condition

$$\sum_{n=m_0+1}^{\infty}|A_n(\tau)\,X_n(x)|<\varepsilon_0$$

for all $\tau \in [0, \infty)$, where ε_0 is an arbitrarily small positive number. Since $\mu_n \to \infty$ as $n \to \infty$, $R_{m_0}(x, \tau)$ is also of a hyperbolic character.

We will use Eq. (8) to determine the criterion for applicability of the hyperbolic equation for description of the thermal state of a layer of thickness l.

Let $\delta_n > 0$ for all n. In this case

$$W^2 > 4\mu_n^2 a^2 \quad \text{for all} \quad n = 1, 2, \ldots, \tag{10}$$

and since $\mu_n \to \infty$ as $n \to \infty$ obviously Eq. (10) is valid at $W = \infty$. Therefore, condition (10) is satisfied within the framework of the phenomenological theory of thermal conductivity. In this case $\nu_1 = -\infty$ and $\nu_2 = -\mu_n^2 a$.

Let $\delta_n > 0$ for all $n \le m_0$. Then

$$W^2 > 4\mu_{m_0}^2 a^2. \tag{11}$$

If the rate of heat propagation in the layer satisfies Eq. (11), then assuming error in ε_0 for all $\tau \in [0, \infty)$, the process may be described by the classical thermal conductivity equation.

Evaluation of Eq. (11) also shows that for any large, but finite heat propagation rate, the hyperbolic component will always occur in the solution of Eqs. (1)-(3). Consequently, the question of the applicability of the classical thermal conductivity equation must be resolved with consideration of the accuracy required in the analysis of the thermal process.

Let $\delta_n < 0$ for all n. Then

$$W^2 < 4\mu_1^2 a^2. \tag{12}$$

If the heat propagation rate satisfies condition (12), then to calculate the temperature field in the layer it is necessary to use the hyperbolic equation. Then $R_n = R_0 \equiv 0$.

For boundary problem I (II) $\mu_n = n\pi/l$, and from Eq. (12) we have an estimate of the geometric dimensions of the region

$$l < \frac{a}{W} \cdot 2\pi,$$

wherein the thermal process is described by the hyperbolic equation.

For metals, e.g., $0 < l < c \cdot 10^{-6}$ m, where the constant c depends on the type of metal. For aluminum, copper, steel, and iron it equals 0.190, 0.243, 0.068, 0.087, respectively. The width of the region for establishment of thermodynamic equilibrium increases markedly for porous bodies. Thus, for cork, $l < 0.177 \cdot 10^{-2}$ m.

Equation (10) indicates that if the process is described by a parabolic equation ($W = \infty$), then the width of the region will coincide with the entire layer.

From Eq. (11) it follows that

$$l > \frac{2\pi m_0 a}{W} \equiv l_1,$$

i.e., to an accuracy of ε_0 the classic thermal conductivity equation can be employed for a layer, the geometric dimensions of which are not smaller than l_1. If $l < l_1$, the hyperbolic equation must be used to describe thermal processes.

For the case of boundary problem (1)-(3) the eigennumbers μ_n of Eq. (5) are defined with the transcendental equation

$$\cot \mu_n l = \left(\mu_n^2 - \frac{\alpha_{12}\alpha_{22}}{\lambda^2}\right)\left(\frac{\mu_n(\alpha_{12} + \alpha_{22})}{\lambda}\right)^{-1}.$$

The transitional component of the problem appears if there exists a number n_0 such that $W = 2\mu_{n_0} a$, i.e., if W satisfies the condition

$$\cot \frac{Wl}{2a} = \frac{\lambda}{\alpha_{12}+\alpha_{22}} \frac{W}{2a} - \frac{\alpha_{12}\alpha_{22}}{\alpha_{12}+\alpha_{22}} \frac{2a}{\lambda W}. \tag{13}$$

Proceeding as in the case of boundary problem I, we obtain: if $W = \infty$, then the process is essentially parabolic; if $W > W_{m_0}$, where $W_{m_0} - m_0$ are the roots (in order of increasing magnitude) of Eq. (13), then to an accuracy of ε the process may be considered parabolic; if $W < W_1$, where $W_1 = \min W_n$, the process is essentially hyperbolic.

We will now determine values of l for the given boundary problem. Since the departure of the eigennumbers of the present problem from those of the first boundary problem can be evaluated in the following manner:

$$\mu_n^{(I)} - \mu_n^{(III)} = \sigma_n,$$

where

$$\sigma_n \in \begin{cases} \left[0,\ \dfrac{\pi}{2l}\right] \text{ for } \mu_n < \lambda^{-1}(\alpha_{12}\alpha_{22})^{1/2}, \\ \left[\dfrac{\pi}{2l},\ \dfrac{\pi}{l}\right] \text{ for } \mu_n > \lambda^{-1}(\alpha_{12}\alpha_{22})^{1/2}; \end{cases}$$

then, using Eq. (11), we obtain the inequality

$$l > \pi m_0 \left(\frac{W}{2a} - \sigma_{m_0}\right)^{-1}, \tag{14}$$

from which it is evident that the classical thermal conductivity equation may be employed for a layer, the dimensions of which are not smaller than the quantity

$$l_3 = \pi m_0 \left(\frac{W}{2a} + \sigma_{m_0}\right)^{-1},$$

with the error thus introduced not exceeding the value of ε_0. If the dimensions of the layer are less than l_3, the hyperbolic equation must be used to study the thermal state of the layer to an accuracy of ε_0.

Let $\varepsilon = \min(\varepsilon_0^{(I)}, \varepsilon_0^{(III)})$ and $m_0 = m_0(\varepsilon)$. In this case, the estimates obtained indicate that the minimum possible (for an accuracy of ε in the classical equation) dimension l_3 for the problem considered is smaller than the minimum possible layer dimension l for boundary problem I by the amount

$$\delta(l) = l_1 - l_3 = \frac{2\pi a}{W} m_0 \frac{2a\sigma_{m_0}}{W + 2a\sigma_{m_0}},$$

while the value of the deviation depends on the heat-exchange conditions on the layer surface and lies within the limits

$$\delta(l) \in \begin{cases} \left[v;\ \dfrac{2\pi a m_0}{W}\left(1 + \dfrac{Wl}{\pi a}\right)^{-1}\right] \text{ for } \mu_{m_0}^{(III)} < \dfrac{(\alpha_{12}\alpha_{22})^{1/2}}{\lambda}, \\ \left[\dfrac{2\pi a m_0}{W}\left(1 + \dfrac{Wl}{\pi a}\right)^{-1};\ \dfrac{2\pi a m_0}{W}\left(1 + \dfrac{Wl}{2\pi a}\right)^{-1}\right] \\ \text{for } \mu_{m_0}^{(III)} > \dfrac{(\alpha_{12}\alpha_{22})^{1/2}}{\lambda}. \end{cases}$$

From inequality (12) we obtain an estimate of the region size for the given boundary problem, where the thermal process in the entire layer is described by a hyperbolic equation

$$l_1 < \pi \left[\frac{W}{2a} + \sigma_1\right], \tag{15}$$

where

$$\sigma_1 \in \begin{cases} \left[0,\ \dfrac{\pi}{2l}\right] \text{ for } \mu_1^{(III)} < (\alpha_{12}\alpha_{22})^{1/2}\lambda^{-1}, \\ \left[\dfrac{\pi}{2l},\ \dfrac{\pi}{l}\right] \text{ for } \mu_1^{(III)} > (\alpha_{12}\alpha_{22})^{1/2}\lambda^{-1}. \end{cases}$$

From Eqs. (14), (15) it follows that in justifying the applicability of either the classical or the hyperbolic equation it is necessary to consider not only the heat propagation rate, but also the character of the interaction between the body under study and the surrounding medium.

LITERATURE CITED

1. A. V. Lykov, Theory of Thermal Conductivity [in Russian], Vysshaya Shkola, Moscow (1967).
2. I. N. Bogaenko, M. V. Boichuk, V. V. Buadze, A. A. Masnev, and Yu. A. Timofeev, Conjugate Heat Exchange Problems in Regions of Complex Configuration [in Russian], Nauka, Tbilisi (1977).
3. E. Kamke, Handbook of Ordinary Differential Equations [in German], Chelsea Publ.